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# Tensor products of prime-power groups

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**Abstract**

We give a bound for the order of the nonabelian tensor product of two prime-power groups. From this we obtain bounds on the third homotopy group of a union of two spaces. We illustrate our bound by using a GAP computer program to determine the order of the nonabelian tensor product $G \otimes H$ for all normal subgroups $G$ and $H$ of the quaternion group of order 32. © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Let $G$ be a finite $p$-group and $H$ a finite $q$-group where the primes $p$ and $q$ are not necessarily distinct. Let there be an action $(g,h) \mapsto {}^g h$ of $G$ on $H$, and an action $(h,g) \mapsto {}^h g$ of $H$ on $G$. The group $G$ is assumed to act on itself by conjugation $(g,g') \mapsto {}^g g' = gg'g^{-1}$, and $H$ is assumed to act on itself similarly. Let us suppose that the various actions are *compatible* in the sense that

$$^{({}^g h)}g' = {}^g({}^h({}^{g^{-1}}g')),$$

$$^{({}^h g)}h' = {}^h({}^g({}^{h^{-1}}h')),$$

for $g, g' \in G,\ h, h' \in H$.

The tensor product $G \otimes H$ was defined in [3] as the group generated by symbols $g \otimes h$ (for $g \in G,\ h \in H$), subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h),$$

$$g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h'),$$

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for $g, g' \in G$, $h, h' \in H$. Homomorphisms

$$\lambda : G \otimes H \to G, \quad g \otimes h \mapsto g\, {}^h g^{-1},$$

$$\mu : G \otimes H \to H, \quad g \otimes h \mapsto {}^g h h^{-1},$$

and an isomorphism

$$G \otimes H \xrightarrow{\cong} H \otimes G, \quad g \otimes h \mapsto h \otimes g,$$

were established in [3].

It was shown in [5] that $G \otimes H$ is a finite $pq$-group. In Section 2 we give a bound for the order of $G \otimes H$. This bound is illustrated in Section 4, where we use a GAP computer program to determine $|G \otimes H|$ for all normal subgroups $G$, $H$ of the quaternion group of order 32. In Section 3 we consider a union of $CW$-spaces $X = A \cup B$; we give a bound on $\pi_3 X$ in terms of the homotopy groups of $A$, $B$ and $A \cap B$, and a certain tensor product.

## 2. Group-theoretic results

If there exists a group $E$ containing both $G$ and $H$ as normal subgroups, then conjugation in $E$ gives rise to compatible actions. In this case we let $GH$ denote the subgroup of $E$ generated by $G$ and $H$, and we define

$$_H G = \{g \in G\text{: there exists an } h \in H \text{ such that } gh^{-1} \in Z(GH)\}.$$

Note that $_H G$ is a normal subgroup of $G$, and that $_H G = G$ if $G \subseteq H$.

We let $\Phi G$ denote the Frattini subgroup $[G, G]G^p$ of $G$. Thus, $G$ is a *d-generator group* if and only if $|G/\Phi G| = p^d$.

**Proposition 1.** *Suppose that G and H are normal subgroups of E, and that actions arise from conjugation in E.*

(i) *If* $p \neq q$ *then* $|G \otimes H| = 1$.

(ii) *Suppose that* $p = q$ *and that G is a d-generator group of order* $p^n$, *H is a* $d'$*-generator group of order* $p^{n'}$, *and* $|_H G|/|_H G \cap \Phi G| = p^k$. *Then*

$$|G \otimes H| \leq p^{nn' - (k+n-d)(n'-d')}.$$

**Proof.** (i) Suppose $p \neq q$. Then $[G, H]$ is trivial since it lies in the intersection of $G$ with $H$. Proposition 2.4 in [3] thus implies an isomorphism

$$G \otimes H \cong G^{ab} \otimes_{\mathbb{Z}} H^{ab},$$

where the right-hand side denotes the usual tensor product of abelian groups. Clearly $G^{ab} \otimes_{\mathbb{Z}} H^{ab}$ is trivial.

(ii) Suppose $p = q$. We shall consider two cases.

*Case* 1: Suppose $[G,H]=1$, and let $H^{ab} \cong C_1 \times C_2 \times \cdots \times C_{d'}$, where $C_i$ denotes a cyclic $p$-group. Proposition 2.4 in [3] yields

$$G \otimes H \cong G^{ab} \otimes H^{ab} \cong (G^{ab} \otimes C_1) \times (G^{ab} \otimes C_2) \times \cdots \times (G^{ab} \otimes C_{d'}).$$

Therefore,

$$\begin{aligned} |G \otimes H| &\leq |G^{ab} \otimes C_1| \times \cdots \times |G^{ab} \otimes C_{d'}| \\ &\leq |G^{ab}|^{d'} \\ &\leq |G|^{d'} \\ &= p^{nd'}. \end{aligned}$$

Since $k \leq d$ we have

$$\begin{aligned} |G \otimes H| &\leq p^{nd'} \\ &= p^{nn'-(d+n-d)(n'-d')} \\ &\leq p^{nn'-(k+n-d)(n'-d')} \end{aligned}$$

as required.

*Case* 2: Suppose $[G,H] \neq 1$. Let $l = n + n'$ with $n \neq 0$, $n' \neq 0$. As an inductive hypothesis, suppose that the proposition has been proved for all $p$-groups $G'$, $H'$ with $|G'||H'| \leq p^t$. This hypothesis is certainly true for $t=2$, since $|G'||H'| \leq p^2$ implies that either $G' \cap H' = 1$ or $G' = H' = C_p$, and consequently, that $[G',H']=1$. Assume the hypothesis true for $t < l$.

A simple exercise, using the class equation, shows that any non-trivial normal subgroup of a finite $p$-group intersects the centre non-trivially. There is thus a central subgroup $N$ of $GH$ such that $N \subseteq [G,H]$ and $|N| = p$. Proposition 9 in [2] easily extends to yield the following lemma.

**Lemma.** *Any central subgroup $Z$ of $GH$ which lies in the intersection $G \cap H$ gives rise to an exact sequence*

$$(G \otimes Z) \times (Z \otimes H) \to G \otimes H \to (G/Z) \otimes (H/Z) \to 1.$$

The inclusion $N \subseteq [G,H]$ implies that the image of the canonical homomorphism ${}_HG \otimes N \to G \otimes H$ is contained in the image of the canonical homomorphism $N \otimes H \to G \otimes H$. (To see this let $g \in {}_HG$ and let $\tau = [g_1,h_1]\dots[g_t,h_t]$ be an element in $N$. Let $\tilde{\tau}$ be an element in $G \otimes H$ which is mapped to $\tau$ by $\lambda$. By the definition of ${}_HG$ there exists an element $h \in H$ such that ${}^hg_i = {}^gg_i$ and ${}^hh_i = {}^gh_i$ for all $i$. Thus ${}^h\tilde{\tau} = {}^g\tilde{\tau}$. Theorem 2.3(d) in [3] implies that, in the tensor product $G \otimes H$, we have

$$g \otimes \tau = {}^g\tilde{\tau}\,\tilde{\tau}^{-1} = {}^h\tilde{\tau}\,\tilde{\tau}^{-1} = (\tau \otimes h)^{-1}.)$$

The lemma with $Z=N$, together with our result on the image of ${}_HG\otimes N\to G\otimes H$, imply an exact sequence

$$(G/{}_HG\otimes N)\times(N\otimes H)\to G\otimes H\to(G/N)\otimes(H/N)\to 1.$$

This sequence and the isomorphism $(G/{}_HG)\otimes N\cong G/({}_HG\Phi G)$ yield

$$\begin{aligned}|G\otimes H| &\le |(G/N)\otimes(H/N)||(G/{}_HG)\otimes N||N\otimes H|\\ &= |(G/N)\otimes(H/N)||G/({}_HG\Phi G)||H/\Phi H|\\ &= |(G/N)\otimes(H/N)|p^{d-k}p^{d'}\\ &\le p^{(n-1)(n'-1)-(k+n-d-1)(n'-d'-1)+d-k+d'}\\ &= p^{nn'-(k+n-d)(n'-d')}\end{aligned}$$

as required. □

When $G=H$, Proposition 1(ii) improves on a bound given by Rocco [12] (see also [6]).

Not all compatible actions arise from conjugation in a group $E$ containing $G$ and $H$. Suppose, for instance, that $G$ and $H$ are cyclic groups of order 8, with generators $x$ and $y$. Compatible actions arise by defining ${}^xy=y^{-1}$, ${}^yx=x^{-1}$. These actions do not arise from conjugation, for if they did we would have $x^2=x^yx^{-1}={}^xyy^{-1}=y^{-2}$, and thus the contradiction $x^2={}^yx^2=x^{-2}$.

So we now work towards a version of Proposition 1 for arbitrary compatible actions. Let us use the action of $H$ on $G$ to form the semi-direct product $G\rtimes H$, in which elements are multiplied according to the rule

$$(g,h)\cdot(g',h')=(g\,{}^hg',hh')$$

for $g,g'\in G$, $h,h'\in H$. Consider the subgroup $P$ of this semi-direct product generated by the elements $(g\,{}^hg^{-1},h\,{}^gh^{-1})$ for $g\in G$, $h\in H$. It is readily verified (or see, for instance, [1]) that $P$ is a normal subgroup of $G\rtimes H$. Following [1], set $G\circ H=(G\rtimes H)/P$. The canonical homomorphisms $G\to G\rtimes H$, $g\mapsto(g,1)$ and $H\to G\rtimes H$, $h\mapsto(1,h)$ induce homomorphisms

$$\partial: G\to G\circ H,$$

$$\delta: H\to G\circ H.$$

One readily verifies (or see [1]) that $\partial$ and $\delta$ are crossed modules. It follows that $ker\,\partial$ is a central subgroup of $G$ which is stable under the action of $H$, and thus that $ker\,\partial$ is a $\mathbb{Z}H$-module. Similarly, $ker\,\delta$ is a $\mathbb{Z}G$-module.

Let $[H,ker\,\partial]_G$ be that central subgroup of $G$ generated by the elements ${}^hxx^{-1}$ for $h\in H$, $x\in ker\,\partial$. Note that $[H,ker\,\partial]_G$ is a $\mathbb{Z}H$-module. We similarly have a $\mathbb{Z}G$-module $[G,ker\,\delta]_H$.

Let us denote by $H_1(G,A)$ the first Eilenberg–Mac Lane homology of $G$ with coefficients in a $G$-module $A$.

**Theorem 2.** (i) *If* $p \neq q$ *then* $G \otimes H \cong [G, ker\,\delta]_H \times [H, ker\,\partial]_G$, *and consequently,*

$$|G \otimes H| = |[G, ker\,\delta]_H||[H, ker\,\partial]_G|.$$

(ii) *Suppose* $p = q$, *and that* $G$ *is a* $d$*-generator group of order* $p^n$, $H$ *is a* $d'$*-generator group of order* $p^{n'}$, *and* $|{}_HG|/|{}_HG \cap \Phi G| = p^k$. *Then*

$$|G \otimes H| \leq K p^{nn' - (k+n-d)(n'-d')},$$

*where* $K = |H_1(G, ker\,\delta)||H_1(H, ker\,\partial)||[H, ker\,\partial]_G||[G, ker\,\delta]_H|$.

**Proof.** Proposition 7 in [2] extends to yield the following generalisation of the lemma in our proof of Proposition 1.

**Lemma.** *Any pair of crossed modules* $\partial : G \to Q$ *and* $\delta : H \to Q$ *yields an exact sequence*

$$(G \otimes ker\,\delta) \times (ker\,\partial \otimes H) \to G \otimes H \to \bar{G} \otimes \bar{H} \to 1, \qquad (*)$$

*where* $\bar{G} = \partial G$, $\bar{H} = \delta H$ *are the images of* $G$, $H$ *in* $Q$.

Taking $Q = G \circ H$, and $\partial$, $\delta$ to be as in the theorem, note that $\bar{G}$, $\bar{H}$ are both normal in $G \circ H$, and so they act on each other by conjugation. Corollary 3.3 in [10] provides an exact sequence

$$1 \to H_1(G, ker\,\delta) \to G \otimes ker\,\delta \xrightarrow{\mu} [G, ker\,\delta]_H \to 1. \qquad (**)$$

(i) Suppose $p \neq q$. Proposition 1(i) tells us that $\bar{G} \otimes \bar{H} = 1$. Since $G$ is a $p$-group and $ker\,\delta$ is a $q$-group, we have $H_1(G, ker\,\delta) = 1$ (see, for instance, [11]). Sequence $(**)$ implies

$$G \otimes ker\,\delta \cong [G, ker\,\delta]_H.$$

Similarly,

$$ker\,\partial \otimes H \cong [H, ker\,\partial]_G.$$

Since the composite homomorphism

$$[G, ker\,\delta]_H \xrightarrow{\cong} G \otimes ker\,\delta \to G \otimes H \to H$$

sends $[G, ker\,\delta]_H$ injectively into $H$, the $q$-group $[G, ker\,\delta]_H$ embeds into $G \otimes H$. Similarly, the $p$-group $[H, ker\,\partial]_G$ embeds into $G \otimes H$. Since $p \neq q$, sequence $(*)$ implies the required isomorphism

$$[G, ker\,\delta]_H \times [H, ker\,\partial]_G \cong G \otimes H.$$

(ii) Suppose $p = q$. The required homomorphism follows from Proposition 1(ii) and the sequences (*) and (**). □

(Note that the isomorphism of Theorem 2(i), in fact, holds for any two finite groups $G$, $H$ with coprime exponents.)

Let $N$ be a normal subgroup in $G$. Using conjugation actions, we can form the tensor product $N \otimes G$. As explained in [3], there is an action of the group G on the tensor product $N \otimes G$ given by

$$^{g}(n \otimes g') = (^{g}n \otimes {}^{g}g')$$

for $g, g' \in G$, $n \in N$. Conversely, an element $\tau \in N \otimes G$ acts on $g \in G$ by $^{\tau}g = (\mu\tau)g(\mu\tau)^{-1}$. These actions are compatible and we can use them to form the tensor product $(N \otimes G) \otimes G$. This construction can be iterated to form the tensor product

$$N \otimes^{c+1} G = (((N \otimes G) \otimes G) \cdots \otimes G)$$

of $N$ with $c$ copies of G. (In this notation, $N \otimes^3 G = (N \otimes G) \otimes G$.) Let us define a central series by

$$\gamma_1(N, G) = N,$$

$$\gamma_{i+1}(N, G) = [\gamma_i(N, G), G].$$

There is a canonical surjection $\mu : N \otimes^c G \twoheadrightarrow \gamma_c(N, G)$ which sends a tensor $(((n \otimes g_1) \otimes g_2) \cdots \otimes g_c)$ to the commutator $[[[n, g_1], g_2], \ldots, g_c]$.

The following corollary is the basis for the main results of [4, 8]. It is also essential to the proof of Theorem 2 in [7].

**Corollary 3.** *Let $N$ be a normal subgroup of a $d$-generator $p$-group $G$. Suppose that $|\gamma_i(N, G)| = p^{m_i}$ for $i = 1, 2, \ldots$. Then, for any $c \geq 1$, we have*

$$|N \otimes^{c+1} G| \leq p^{m_c d + m_{c-1} d^2 + \cdots + m_1 d^c}.$$

**Proof.** For $c = 1$ the corollary follows from Proposition 1(ii). For arbitrary $i \geq 1$ let us define

$$J_i(N, G) = ker(\mu : N \otimes^i G \to \gamma_i(N, G)).$$

For $c \geq 2$, there is thus an exact sequence

$$J_c(N, G) \otimes G \to (N \otimes^c G) \otimes G \to \gamma_c(N, G) \otimes G \to 1. \qquad (***)$$

Now, $G$ and $J_c(N, G)$ act trivially on each other. Thus, $|J_c(N, G) \otimes G| \leq |J_c(N, G)|^d \leq |N \otimes^c G|$. Also, Proposition 1(ii) implies $|\gamma_c(N, G) \otimes G| \leq p^{m_c d}$. So sequence (***) provides the recurrence relation

$$|N \otimes^{c+1} G| \leq p^{m_c d} |N \otimes {}^c G|$$

from which the corollary follows. (Note that this proof does not use the finiteness of $G$.) □

## 3. A topological application

Suppose that a $CW$-space $X$ is a union $X = A \cup B$ of two path-connected $CW$-subspaces $A$ and $B$ whose intersection $C = A \cap B$ is path-connected. Some of the homotopy structure of the space $X$ can be calculated in terms of the homotopy structure of the spaces $A, B$ and $C$. For instance, van Kampen's theorem on the fundamental group describes $\pi_1 X$ as an amalgamated sum of groups:

$$\pi_1 X \cong \pi_1 A *_{\pi_1 C} \pi_1 B.$$

A "two-dimensional analogue" of van Kampen's Theorem is used in [1] to describe the second relative homotopy group $\pi_2(X, C)$ under the hypothesis that the canonical homomorphisms $\pi_1 C \to \pi_1 A$, $\pi_1 C \to \pi_1 B$ are surjective:

$$\pi_2(X, C) \cong \pi_2(A, C) \circ \pi_2(B, C).$$

Here the symbol $\circ$ denotes the construction of the previous section, the groups $\pi_2(A, C)$, $\pi_2(B, C)$ acting on one another via the boundary homomorphisms $\pi_2(A, C) \to \pi_1 C$, $\pi_2(B, C) \to \pi_1 C$ and actions of $\pi_1 C$.

A "three-dimensional analogue" of van Kampen's theorem is used in [3] to describe the triad homotopy group $\pi_3(X, A, B)$ under the hypothesis that the canonical homomorphisms $\pi_1 C \to \pi_1 A$, $\pi_1 C \to \pi_1 B$ are surjective:

$$\pi_3(X, A, B) \cong \pi_2(A, C) \otimes \pi_2(B, C).$$

Using the exact sequences (for $n \geq 1$) (see [13])

$$\to \pi_n C \to \pi_n A \to \pi_n(A, C) \to \pi_{n-1} C \to,$$

$$\to \pi_n C \to \pi_n B \to \pi_n(B, C) \to \pi_{n-1} C \to,$$

$$\to \pi_n(B, C) \to \pi_n(X, A) \to \pi_n(X, A, B) \to \pi_{n-1}(B, C) \to,$$

one readily obtains the following bound on $\pi_3 X$.

**Proposition 5.** *Suppose that the canonical homomorphisms $\pi_1 C \to \pi_1 A, \pi_1 C \to \pi_1 B$ are surjective. Then*

$$|\pi_3 X| \leq \frac{|\pi_1 A| \cdot |\pi_3 A|}{|\pi_2 A|} \cdot \frac{|\pi_1 B| \cdot |\pi_3 B|}{|\pi_2 B|} \cdot \frac{|\pi_2 C|^2}{|\pi_1 C|^2} \cdot |\pi_2(A, C) \circ \pi_2(B, C)| \cdot |\pi_2(A, C) \otimes \pi_2(B, C)|.$$

*The bound is attained if, for instance, the homotopy groups $\pi_3 A$, $\pi_3 B$, $\pi_2 C$ are all trivial.*

We clearly have

$$|\pi_2(A,C) \circ \pi_2(B,C)| \leq |\pi_2(A,C)||\pi_2(B,C)|.$$

Thus, Theorem 2 yields an explicit bound on $|\pi_3 X|$ in the case where $\pi_2(A,C)$ and $\pi_2(B,C)$ are known prime-power groups.

## 4. Computer computations

We now consider all pairs of normal subgroups of the quaternion group, $Q_{16} = \langle a,b | a^{16} = b^8 a^2 = ab^{-1}ab = 1 \rangle$, with actions being conjugations in $Q_{16}$. Twelve such subgroups exist. Table 1 presents the different actions which arise in this way, by exhibiting the images of the generators of $H$ under the actions of the generators of $G$.

Table 2 lists $|G \otimes H|$ for all pairs of normal subgroups $G$ and $H$ in $Q_{16}$. Since $G \otimes H \cong H \otimes G$ the table only includes the case $|G| \leq |H|$. We consider two pairs $(G,H)$ and $(G',H')$ to be isomorphic if there is an isomorphism $\phi : GH \to G'H'$ that restricts to isomorphisms $G \xrightarrow{\cong} G'$, $H \xrightarrow{\cong} H'$. Since isomorphic pairs yield isomorphic tensor products, the table contains just one entry for each isomorphism class of pairs. There is a certain asymmetry in the bound of Theorem 2, yet $G \otimes H \cong H \otimes G$. Thus,

Table 1
The conjugation actions of subgroups $G$ on subgroups $H$

| | | G | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Generators | 1 | $a^2$ | $a$ | $b^4$ | $a^2$ | $ab^4$ | $a$ | $b^4$ | $a^2b^2$ | $ab^2$ | $a$ | $b^2$ | $b$ | $ab$ | $a$ | $b$ |
| | $x=a^2$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ |
| | $x=a$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x^3$ | $x^3$ | $x$ | $x^3$ |
| | $x=b^4$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ |
| | $x=a^2$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ |
| | $y=ab^4$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $xy$ | $xy$ | $y$ | $xy$ |
| | $x=a$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x^3$ | $x^3$ | $x$ | $x^3$ |
| | $y=b^4$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ |
| H | $x=a^2b^2$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ |
| | $x=ab^2$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x^5$ | $x^5$ | $x$ | $x^5$ |
| | $x=a$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x^3$ | $x^3$ | $x$ | $x^3$ |
| | $y=b^2$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ |
| | $x=b$ | $x$ | $x$ | $x^9$ | $x$ | $x$ | $x^9$ | $x^9$ | $x$ | $x$ | $x^9$ | $x^9$ | $x$ | $x$ | $x^9$ | $x^9$ | $x$ |
| | $x=ab$ | $x$ | $x$ | $x^9$ | $x$ | $x$ | $x^9$ | $x^9$ | $x$ | $x$ | $x^9$ | $x^9$ | $x$ | $x^9$ | $x$ | $x^9$ | $x^9$ |
| | $x=a$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x^3$ | $x^3$ | $x$ | $x^3$ |
| | $y=b$ | $y$ | $y$ | $x^2y$ | $y$ | $y$ | $x^2y$ | $x^2y$ | $y$ | $y$ | $x^2y$ | $x^2y$ | $y$ | $y$ | $x^2y$ | $x^2y$ | $y$ |

Table 2
$|G \otimes H|$ for $G, H \trianglelefteq Q_{16}$

| Generators $G$ | $\|G\|$ | Generators $H$ | $\|H\|$ | $\log_2(\|G\otimes H\|)$ | $\log_2$ (bound of Theorem 2) |
|---|---|---|---|---|---|
| $a^2$ | 2 | $a^2$ | 2 | 1 | 1 |
| $a^2$ | 2 | $a$ | 4 | 1 | 1 |
| $a^2$ | 2 | $a^2, ab^4$ | 4 | 2 | 2 |
| $a^2$ | 2 | $a, b^4$ | 8 | 2 | 2 |
| $a^2$ | 2 | $a^2b^2$ | 8 | 1 | 1 |
| $a^2$ | 2 | $a, b^2$ | 16 | 2 | 2 |
| $a^2$ | 2 | $b$ | 16 | 1 | 1 |
| $a^2$ | 2 | $a, b$ | 32 | 2 | 2 |
| $a$ | 4 | $a$ | 4 | 2 | 2 |
| $a$ | 4 | $a^2, ab^4$ | 4 | 2 | 4 |
| $a$ | 4 | $a, b^4$ | 8 | 3 | 4 |
| $a$ | 4 | $a^2b^2$ | 8 | 2 | 2 |
| $a$ | 4 | $a, b^2$ | 16 | 3 | 4 |
| $a$ | 4 | $b$ | 16 | 2 | 5 |
| $a$ | 4 | $a, b$ | 32 | 3 | 4 |
| $b^4$ | 4 | $b$ | 16 | 2 | 2 |
| $b^4$ | 4 | $a, b$ | 32 | 3 | 4 |
| $a^2, ab^4$ | 4 | $a^2, ab^4$ | 4 | 4 | 4 |
| $a^2, ab^4$ | 4 | $a, b^4$ | 8 | 4 | 4 |
| $a^2, ab^4$ | 4 | $a^2b^2$ | 8 | 2 | 2 |
| $a^2, ab^4$ | 4 | $a, b^2$ | 16 | 4 | 4 |
| $a^2, ab^4$ | 4 | $b$ | 16 | 2 | 5 |
| $a^2, ab^4$ | 4 | $a, b$ | 32 | 3 | 4 |
| $a, b^4$ | 8 | $a, b^4$ | 8 | 5 | 6 |
| $a, b^4$ | 8 | $a^2b^2$ | 8 | 3 | 3 |
| $a, b^4$ | 8 | $a, b^2$ | 16 | 5 | 6 |
| $a, b^4$ | 8 | $b$ | 16 | 3 | 6 |
| $a, b^4$ | 8 | $a, b$ | 32 | 2 | 6 |
| $a^2b^2$ | 8 | $a^2b^2$ | 8 | 3 | 3 |
| $a^2b^2$ | 8 | $a, b^2$ | 16 | 4 | 6 |
| $a^2b^2$ | 8 | $b$ | 16 | 3 | 3 |
| $a^2b^2$ | 8 | $a, b$ | 32 | 4 | 6 |
| $ab^2$ | 8 | $b$ | 16 | 3 | 6 |
| $ab^2$ | 8 | $a, b$ | 32 | 4 | 6 |
| $a, b^2$ | 16 | $a, b^2$ | 16 | 6 | 8 |
| $a, b^2$ | 16 | $b$ | 16 | 4 | 7 |
| $a, b^2$ | 16 | $a, b$ | 32 | 2 | 8 |
| $b$ | 16 | $b$ | 16 | 4 | 4 |
| $b$ | 16 | $ab$ | 16 | 4 | 7 |
| $b$ | 16 | $a, b$ | 32 | 5 | 8 |
| $ab$ | 16 | $a, b$ | 32 | 5 | 8 |
| $a, b$ | 32 | $a, b$ | 32 | 7 | 10 |

a lower bound may sometimes be obtained by interchanging the roles of $G$ and $H$. However, the table does not involve such interchanges.

Table 2 was computed using a GAP program based on the algorithm described in [9]. Details of the program may be obtained by e-mailing the authors at graham.ellis@ucg.ie.

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